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$\mathbb{R}^{2n+1}$ is a universal contact manifold for reduction

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Abstract. We show that all contact manifolds can be obtained by reduction from a universal contact manifold $\mathbb{R}^{2N+1}$. We also prove an equivariant version and discuss the relationship with the corresponding results for symplectic manifolds.

1. Introduction

Symplectic manifolds are the natural geometrical setting for time-independent mechanics; for time-dependent mechanics attention focuses on contact and cosymplectic manifolds. As is well known, (symplectic) reduction is a powerful tool to investigate complicated mechanical systems. It is thus natural to investigate such a procedure also for contact and cosymplectic manifolds. In [2] a version of the Marsden–Weinstein symplectic reduction theorem for contact and cosymplectic manifolds has been proven. More recently in [6] a more general reduction procedure for contact manifolds (in a wider sense using foliations/ideals of (contact) forms) has been introduced: reduction by a closed submanifold C; the author characterized such a reduction by the geometry of its graph and of the submanifold C. In [9] the even more general reduction of Jacobi manifolds has been discussed.

By Darboux's theorem, $\mathbb{R}^{2N}$ with its canonical symplectic form is the local model for any symplectic manifold [1]. In [7, 8] it has been shown that it is also a universal model in the context of reduction, i.e. any symplectic manifold can be obtained by reduction from some $\mathbb{R}^{2N}$. Recently this result has been extended to cosymplectic manifolds [11]. However, in that case the local model, i.e. $\mathbb{R}^{2N+1} \cong \mathbb{R} \times T^*\mathbb{R}^N$ with its canonical cosymplectic structure, cannot serve as universal model. In fact, a universal model for cosymplectic manifolds in the context of reduction is of the form $\mathbb{R} \times T^*(\mathbb{R}^N \times \mathbb{T}^k)$.

The aim of this paper is to discuss the case of contact manifolds. Although contact and cosymplectic manifolds have the same $\mathbb{R}^{2n+1}$ as local support, they have different local structures. In local coordinates $(t, q^1, \ldots, q^n, p_1, \ldots, p_n)$ the canonical contact form η_n is given by $\eta_n = \mathrm{d}t + \sum_{i=1}^n p_i \, \mathrm{d}q^i$, and the canonical cosymplectic structure (η, Ω) is given by the forms $\eta = \mathrm{d}t$ and $\Omega = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}p_i$. Since they have different local models,

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it should not come as a surprise that they also have different universal models. In fact, we prove that every contact manifold can be obtained by reduction from some $\mathbb{R}^{2N+1}$ with its canonical contact form. We also prove an equivariant version of this result. Finally, we consider the case of regular contact manifolds and discuss the relationship between the universal models for contact and symplectic manifolds.

2. Contact manifolds and their reduction

A *contact manifold* is a pair (M, η) in which M is an odd, say $(2n + 1)$, dimensional manifold and η a 1-form on M, called the *contact 1-form*, such that $\eta \wedge (\mathrm{d}\eta)^n$ is nowhere zero (see [2, 3, 5, 10, 12]). If for an arbitrary 1-form η we denote by $b\colon TM \to T^*M$ the vector bundle morphism

$$X \in T_xM \mapsto b(X) = \iota(X)(\mathrm{d}\eta)_x + (\iota(X)\eta_x) \cdot \eta_x \in T_x^*M$$

then η is a contact 1-form if and only if b is a vector bundle isomorphism (see [2]). If (M, η) is a contact manifold then the vector field $\mathcal{R} = b^{-1}(\eta)$ is the unique vector field on M such that $\iota(\mathcal{R})\eta = 1$ and $\iota(\mathcal{R})\,\mathrm{d}\eta = 0$; $\mathcal{R}$ is called the *Reeb vector field* of M. Two contact manifolds (M_1, η_1) and (M_2, η_2) are said to be *isomorphic* if there is a diffeomorphism $F\colon M_1 \to M_2$ such that $F^*\eta_2 = \eta_1$ (and thus $F_*\mathcal{R}_1 = \mathcal{R}_2$).

The canonical example of a contact manifold is given by the product space $\mathbb{R} \times T^*Q$ with Q an arbitrary manifold, endowed with the 1-form η_Q defined by $\eta_Q = \mathrm{d}t + \theta_Q$, where t is the coordinate on $\mathbb{R}$ and θ_Q the canonical 1-form on T^*Q. In particular, if $(q^1, \ldots, q^n)$ are local coordinates on Q, then η_Q is given by

$$\eta_Q = \mathrm{d}t + \sum_{i=1}^{n} p_i\,\mathrm{d}q^i \tag{1}$$

where the p_i are the associated coordinates in the fibres of T^*Q. In fact, this expression is the local model for any contact manifold: a Darboux theorem (see [2, 12]) states that around each point m of a contact manifold (M, η) there exist local coordinates $(t, q^1, \ldots, q^n, p_1, \ldots, p_n)$, called *Darboux coordinates*, such that η has the form (1). In Darboux coordinates the Reeb vector field obviously has the form $\mathcal{R} = \partial_t$.

In order to describe the contact reduction procedure, suppose that (M, η) is a contact manifold and suppose that $C \subset M$ is a submanifold satisfying the following conditions:

(i) $\mathcal{R}$ is tangent to C;

(ii) the characteristic distribution $\mathcal{F} = \ker \eta_{|C} \cap \ker \mathrm{d}\eta_{|C}$ has constant rank on C, i.e. $\mathcal{F}$ is a foliation on C;

(iii) the space of leaves $M_r = C/\mathcal{F}$ is a differentiable manifold and the canonical projection $\pi\colon C \to M_r$ is a submersion.

If these conditions are satisfied, standard arguments show that there exists a unique 1-form η_r on M_r such that $\pi^*\eta_r = \eta_{|C}$. Furthermore, it is not hard to show that the couple (M_r, η_r) is a contact manifold whose Reeb vector field $\mathcal{R}_r$ is given by $\mathcal{R}_r = \pi_*\mathcal{R}_{|C}$. In these circumstances we will say that *the contact manifold* (M_r, η_r) *is the reduction of* (M, η) *by the submanifold* C. Moreover, the reader can easily establish the following propositions.

Proposition 2.1. If a contact manifold (M_2, η_2) is the reduction of a contact manifold (M_1, η_1) by the submanifold $C_1 \subset M_1$, and if (M_1, η_1) is the reduction of a contact manifold (M_0, η_0) by the submanifold $C_0 \subset M_0$, then (M_2, η_2) is the reduction of (M_0, η_0) by the submanifold $C_2 = \pi_0^{-1}(C_1)$, where $\pi_0\colon C_0 \to M_1$ denotes the canonical projection.

Proposition 2.2. Let (M_r, η_r) be the reduction of the contact manifold (M, η) by the submanifold C. If the Reeb vector field $\mathcal{R}$ is complete on M and if C is closed in M, then the Reeb vector field $\mathcal{R}_r$ of (M_r, η_r) is complete too.

3. The universal model

The Darboux theorem tells us that $\mathbb{R}^{2n+1} \equiv \mathbb{R} \times T^*\mathbb{R}^n$ with the 1-form $\eta_n = \mathrm{d}t + \sum_i p_i\, \mathrm{d}q^i$ is a local model for any contact manifold. The next proposition tells us that it is also the universal model as far as reduction of contact manifolds is considered.

Theorem 3.1. Let (M, η) be a contact manifold. Then there exists an integer N and a submanifold $C \subset \mathbb{R}^{2N+1}$ such that (M, η) is the contact reduction of $(\mathbb{R}^{2N+1}, \eta_N)$ by the submanifold C.

Proof. The idea of the proof is to break the reduction into two reductions, which are concatenated at the end by proposition 2.1.

For the first step we define $M_1 = \mathbb{R} \times T^*M$ with its contact form $\eta_1 \equiv \eta_M = \mathrm{d}t + \theta_M$; as said before, its Reeb vector field is given by $\mathcal{R}_1 = \partial_t$. Now let $D \subset \mathbb{R} \times M$ be the domain of the flow of the vector field $\mathcal{R}$ on M, and let $f\colon D \to M_1$ be the embedding

$$f(t, m) = (t, \eta_m) \in \mathbb{R} \times T^*_m M.$$

In other words, f is a diffeomorphism between D and the submanifold $C_1 = f(D) \subset M_1$. Obviously $\mathcal{R}_1$ is tangent to C_1, and thus we can try to perform contact reduction with respect to C_1. Since f is a diffeomorphism, we perform the computations in D. Now $f^*\eta_1 = \mathrm{d}t + \eta$, and thus the characteristic distribution $\ker(f^*\eta_1) \cap \ker(f^*\,\mathrm{d}\eta_1)$ is spanned by the vector field $\mathcal{R} - \partial_t$. By definition of D, each integral curve of this vector field intersects $\{0\} \times M \equiv M$ in exactly one point. Since the restriction of $f^*\eta_1$ to $\{0\} \times M$ equals η, we conclude that (M, η) is the contact reduction of (M_1, η_1) by the submanifold C_1.

For the second step we invoke Whitney's embedding theorem to find an embedding $i\colon M \to \mathbb{R}^N$ for some integer N. We then define $M_2 = \mathbb{R} \times T^*\mathbb{R}^N$ with its contact form $\eta_2 = \eta_N$. Denoting by $\pi\colon T^*\mathbb{R}^N \to \mathbb{R}^N$ the canonical projection, we define the submanifold $C_2 = \mathbb{R} \times \pi^{-1}(i(M)) \subset M_2$, with the projection $p\colon C_2 \to M_1$ defined by

$$p(t, \beta_{i(m)}) = (t, i^*\beta_{i(m)}).$$

This is well defined because each point of C_2 is by definition of the form $(t, \beta_{i(m)})$. A direct computation yields the equality $p^*\eta_1 = (\eta_2)_{|C_2}$, from which we deduce (using that η_1 is a contact form on M_1) that $\ker(Tp)$ equals $\ker((\eta_2)_{|C_2}) \cap \ker((\mathrm{d}\eta_2)_{|C_2})$. Since the fibres of p are connected, we conclude that M_1 is the quotient of C_2 by the characteristic foliation, and thus that (M_1, η_1) is the contact reduction of (M_2, η_2) by the submanifold C_2.

We finish the proof by invoking proposition 2.1 to create out of these two reductions a single contact reduction. □

Remark 3.2. Although the Reeb vector field of $(\mathbb{R}^{2N+1}, \eta_N)$, which is ∂_t, is complete, the Reeb vector field of a general contact manifold need not be complete. It thus follows from proposition 2.2 that we cannot always take the submanifold C in theorem 3.1 to be closed. In the proof of theorem 3.1 this is reflected by the fact that the domain D of the flow of $\mathcal{R}$ need not be the whole of $\mathbb{R} \times M$.

We now turn our attention to contact manifolds with symmetry. A smooth action of a Lie group G on a contact manifold (M, η) is said to be a *contact action* if it preserves the contact form η, i.e. for all $g \in G$ we have $g^*\eta = \eta$. One also says that G *is a symmetry group of the contact manifold* (M, η). If (M_r, η_r) is the contact reduction of (M, η) by the submanifold $C \subset M$, and if this C is invariant under the action of G, we have a natural induced smooth action of G on M_r. Moreover, this induced action is a contact action. In such a case we will say that (M_r, η_r, G) *is the equivariant contact reduction of* (M, η, G) *by the (invariant) submanifold* C. Under certain hypotheses our universal model remains valid in the equivariant context.

Theorem 3.3. Let G act smoothly on a contact manifold (M, η). Suppose that the action is a contact action, that G is compact and that M is of finite type. Then there exists an integer N, an orthogonal action of G on $\mathbb{R}^N$ with a canonically induced contact action on $\mathbb{R} \times T^*\mathbb{R}^N$, and an invariant submanifold $C \subset \mathbb{R}^{2N+1}$ such that (M, η, G) is the equivariant contact reduction of $(\mathbb{R}^{2N+1}, \eta_N, G)$ by C.

Proof. We follow the steps of the proof of theorem 3.1. For $g \in G$ we define the action on $M_1 = \mathbb{R} \times T^*M$ by $g(t, \beta_m) = (t, (g^{-1})^*\beta_m) \in \mathbb{R} \times T^*_{g(m)}M$. It is standard that this action on M_1 is a contact action. Moreover, since η and thus $\mathcal{R}$ are invariant under G, the submanifold C_1 is invariant under the G-action on M_1. It thus remains to prove that the induced action on the contact reduction $M_r = M$ coincides with the initial action. But this follows easily once we let G act trivially on $\mathbb{R}$ and realize that then the embedding $f: D \to M_1$ is equivariant. We conclude that (M, η, G) is the equivariant contact reduction of (M_1, η_1, G) by the submanifold C_1.

For the second step we use our hypotheses. Since G is compact and M is of finite type, we can invoke the Mostow–Palais theorem (see [13, 14, 4, pp 173, 218]) to conclude that there exist (i) an integer N, (ii) an orthogonal action of G on $\mathbb{R}^N$ and (iii) an equivariant embedding $i: M \to \mathbb{R}^N$. As in the first step above, we lift the action of G on $\mathbb{R}^N$ to a contact action on $\mathbb{R} \times T^*\mathbb{R}^N$. It now is easy to show that C_2 is invariant under this G-action, and that the projection $p: C_2 \to M_1$ is equivariant. We conclude that (M_1, η_1, G) is the equivariant contact reduction of $(\mathbb{R}^{2N+1}, \eta_N, G)$ by the submanifold C_2.

The proof is finished once the reader ascertains that proposition 2.1 is also true in the equivariant case. □

4. Regular contact reduction and symplectic reduction

In this section we want to compare contact reduction with symplectic reduction, which is the following procedure. Let (V, ω) be a symplectic manifold, and let $B \subset V$ be a submanifold such that $\mathcal{F} = \ker \omega_{|\mathcal{B}}$ has constant rank (on B). If the space of leaves $V_r = B/\mathcal{F}$ admits the structure of a manifold such that the canonical projection $\pi: B \to V_r$ is a submersion, then there exists a unique symplectic form ω_r on V_r such that $\pi^*\omega_r = \omega_{|B}$. In such a case one says that (V_r, ω_r) is the *symplectic reduction of* (M, ω) *by the submanifold* B. In [7] and [8] it has been shown that $T^*\mathbb{R}^N$ is a universal model for (equivariant) symplectic reduction.

Now let (M, η) be a contact manifold. If the space V of orbits of the Reeb vector field $\mathcal{R}$, $V = M/\mathcal{R}$, admits the structure of a manifold such that the canonical projection $\pi: M \to V$ is a submersion, then there exists a unique symplectic form ω on V such that $\pi^*\omega = \mathrm{d}\eta$. In such a case one says that (M, η) *is a regular contact manifold* (see [3, 5, 15]).

If the Reeb vector field $\mathcal{R}$ is complete, we can define the period function λ on M by

$$\lambda(m) = \inf_{t>0, \phi_t(m)=m} t$$

where ϕ_t denotes the flow of $\mathcal{R}$. Since $\mathcal{R}$ vanishes nowhere, it follows that $\lambda(m)$ is strictly positive (possibly infinite). Using the results of [15] (see also [3] and [5]), one can show the following result.

Proposition 4.1. Let (M, η) be a regular contact manifold with complete Reeb vector field $\mathcal{R}$. Then the period function λ is constant on M and we have the two following cases.

(i) If $\lambda(m) \equiv c < \infty$, then M is a principal fibre bundle over $V = M/\mathcal{R}$ with group S^1; the 1-form η is a connection form on this bundle with curvature form Ω. Moreover, the 2-form Ω/c determines an integral cohomology class on V, which is the characteristic class of the circle bundle $M \to V$.

(ii) If $\lambda \equiv \infty$, then M is a principal fibre bundle over $V = M/\mathcal{R}$ with group $\mathbb{R}$; the 1-form η is a connection in this bundle with curvature form Ω.

Given a regular contact manifold (M, η) and a contact reduction (M_r, η_r) of (M, η) by a submanifold $C \subset M$, it is tempting to claim that (M_r, η_r) is a regular contact manifold and that $V_r = M_r/\mathcal{R}_r$ is the symplectic reduction of $V = M/\mathcal{R}$ by the submanifold $B = C/\mathcal{R}$. However, apart from the technical problem that B need not be a submanifold of V, a moment's thought will show that theorem 3.1 tells us that either all contact manifolds are regular, or the claim is false, simply because the universal model is obviously regular. In order to refute the claim, it thus suffices to exhibit a non-regular contact manifold (M, η).

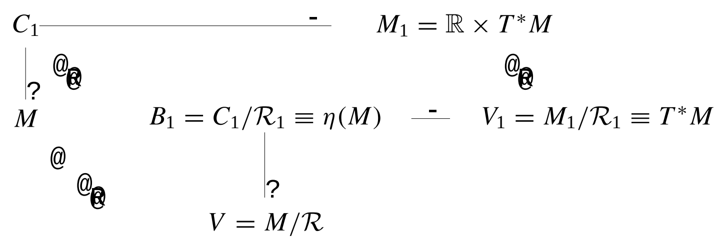
Example 4.2. (See also [3].) Let M be the three-dimensional torus $\mathbb{T}^3 \equiv (\mathbb{R}/2\pi\mathbb{Z})^3$ with the contact form $\eta = \cos(z)\,\mathrm{d}x + \sin(z)\,\mathrm{d}y$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$, confounded with the induced (cyclic) coordinates on $\mathbb{T}^3$. The Reeb vector field is given by

$$\mathcal{R} = \cos(z)\partial_x + \sin(z)\partial_y$$

which is complete because $\mathbb{T}^3$ is compact. For fixed z, the integral curves of $\mathcal{R}$ are 'straight lines' in $\mathbb{T}^2$ with slope $\alpha = \tan(z)$. There are now two ways to see that this contact manifold is not regular. Either by noting that for irrational α the quotient is not a manifold, or by noting that the period function λ is not constant on M and to conclude by applying proposition 4.1.

Now recall that theorem 3.1 says that every contact manifold, and thus in particular every regular contact manifold, can be obtained by reduction from a regular universal contact manifold. But the proof of theorem 3.1 consists of two (concatenated) steps. It is thus interesting to note that in the regular case even the intermediate result is regular.

Lemma 4.3. Let (M, η) be a regular contact manifold with Reeb vector field $\mathcal{R}$. Then $(M_1 = \mathbb{R} \times T^*M, \eta_1 = \mathrm{d}t + \theta_M)$ is a regular contact manifold for which we have the commutative diagram:



Here C_1 is an open subset of $\mathbb{R} \times \eta(M) \subset \mathbb{R} \times T^*M$, which is the whole of $\mathbb{R} \times \eta(M)$ in case $\mathcal{R}$ is complete. In particular, the symplectic manifold (V, ω) is the symplectic reduction of the (canonical) cotangent bundle $(T^*M, \mathrm{d}\theta_M)$ by the submanifold $\eta(M)$.

Proof. That V_1 equals T^*M and that B_1 equals $\eta(M)$ follows immediately from the proof of theorem 3.1. To show that V is the symplectic reduction of V_1 by B_1 we note that $\eta\colon M \to B_1 \subset T^*M$ is a diffeomorphism. Since $\mathrm{d}\theta_M$ is the canonical symplectic form on T^*M, it follows easily that $\eta^*((\mathrm{d}\theta_M)_{|B_1}) = \mathrm{d}\eta$, and thus $(B_1, (\mathrm{d}\theta_M)_{|B_1})$ is isomorphic to $(M, \mathrm{d}\eta)$. Since $\ker(\mathrm{d}\eta)$ is generated by the Reeb vector field $\mathcal{R}$, the result follows. □

Remark 4.4. If in lemma 4.3 the vector field $\mathcal{R}$ is complete, the announced result can also be deduced from proposition 4.1 and the constructions used in [7] (in case $\lambda = \infty$ a slight modification is needed).

Combining the results of lemma 4.3 with the second step in the proof of theorem 3.1, one can easily prove the following result.

Theorem 4.5. Let (M, η) be a regular contact manifold with Reeb vector field $\mathcal{R}$. Then there exist an integer N, a submanifold B of $T^*\mathbb{R}^N \equiv \mathbb{R}^{2N}$, and an open subset C of $\mathbb{R} \times B \subset \mathbb{R} \times T^*\mathbb{R}^N$ with the following properties.

(i) The symplectic manifold $V = M/\mathcal{R}$ is the symplectic reduction of the (canonical) cotangent bundle $T^*\mathbb{R}^N$ by the submanifold B.

(ii) The (regular) contact manifold M is the contact reduction of $\mathbb{R} \times T^*\mathbb{R}^N$ by the submanifold C.

(iii) If $\mathcal{R}$ is complete, C is the whole of $\mathbb{R} \times B$.

We now turn our attention to regular contact manifolds (M, η) with symmetry. If G acts smoothly on M preserving η, it also preserves the Reeb vector field $\mathcal{R}$. It then follows that there exists an induced smooth action of G on $V = M/\mathcal{R}$ which preserves the symplectic form ω on V induced by $\mathrm{d}\eta$. In [8] it has been shown that the existence of a momentum map is a necessary condition for the existence of a universal equivariant model for symplectic manifolds (simply because the universal model $\mathbb{R}^{2N}$ always allows a momentum map, and the existence of a momentum map is preserved by the reduction procedure). We now show that, if the symplectic manifold is deduced from a regular contact manifold, then there automatically exists a momentum map.

Lemma 4.6. Let (M, η) be a regular contact manifold with Reeb vector field $\mathcal{R}$ and let G be a symmetry group of (M, η). Denote by π the canonical projection $\pi\colon M \to V = M/\mathcal{R}$, and by $\mathfrak{g}$ the Lie algebra of G. Then the map $J\colon V \to \mathfrak{g}^*$ defined by

$$\langle J(\pi(m)), X\rangle = \eta_m(X_M(m))$$

is an Ad*-equivariant momentum map for the induced symplectic action of G on V. Here X_M denotes the fundamental vector field on M associated to $X \in \mathfrak{g}$.

Proof. Since $\mathcal{L}(\mathcal{R})\eta = 0$ by definition of $\mathcal{R}$ and since $\mathcal{L}(\mathcal{R})X_M = -\mathcal{L}(X_M)\mathcal{R} = 0$ because the G-action preserves $\mathcal{R}$, it follows that $\mathcal{L}(\mathcal{R})[\eta(X_M)] = 0$. This proves that J is well defined on V.

To prove that J is a momentum map, we introduce, for $X \in \mathfrak{g}$, the function $J_X : V \to \mathbb{R}$, $v \mapsto \langle J(v), X\rangle$. We then have to prove $\iota(X_V)\omega + \mathrm{d}J_X = 0$, where $X_V = \pi_* X_M$ is the fundamental vector field on V associated to $X \in \mathfrak{g}$. Since π^* is injective, it is sufficient to prove that $\pi^*(\iota(X_V)\omega + \mathrm{d}J_X) = 0$. But this follows from the following calculation

$$\pi^*(\iota(X_V)\omega + \mathrm{d}J_X) = \iota(X_M)\mathrm{d}\eta + \mathrm{d}(\iota(X_M)\eta) = \mathcal{L}(X_M)\eta = 0$$

where the last conclusion follows from the invariance of η.

Finally, to prove equivariance, we compute

$$\begin{aligned}\langle J(g\pi(m)), X\rangle &= \langle J(\pi(gm)), X\rangle = \eta_{gm}(X_M(gm)) = (g^*\eta)_m(((\mathrm{Ad}(g^{-1})X)_M)(m)) \\ &= \langle J(\pi(m)), \mathrm{Ad}(g^{-1})X\rangle = \langle Ad^*(g)J(\pi(m)), X\rangle\end{aligned}$$

where we have used the fact that η is invariant and the formula $g_* X_M = (\mathrm{Ad}(g)X)_M$, valid for arbitrary fundamental vector fields. □

We now state the equivariant versions of lemma 4.3 and theorem 4.5; their proofs, which are left to the reader, are straightforward adaptions of the non-equivariant versions and the results of [8].

Lemma 4.7. Let (M, η) be a regular contact manifold and let G be a symmetry group of (M, η). Then the cotangent lift of the action of G on M to T^*M, combined with the trivial action on $\mathbb{R}$, turns G into a symmetry group of the contact manifold $\mathbb{R} \times T^*M$. Moreover, the diagram in lemma 4.3 is equivariant for these actions (together with the induced actions on T^*M and V).

Theorem 4.8. Let (M, η) be a regular contact manifold of finite type and let G be a compact symmetry group of (M, η). Then there exist an integer N, a submanifold B of $T^*\mathbb{R}^N \equiv \mathbb{R}^{2N}$, an open subset C of $\mathbb{R} \times B \subset \mathbb{R} \times T^*\mathbb{R}^N$ and an orthogonal action of G on $\mathbb{R}^N$ with the following properties.

(i) The symplectic manifold $V = M/\mathcal{R}$ is the equivariant symplectic reduction of the (canonical) cotangent bundle $T^*\mathbb{R}^N$ by the submanifold B.

(ii) The (regular) contact manifold M is the equivariant contact reduction of $\mathbb{R} \times T^*\mathbb{R}^N$ by the submanifold C.

(iii) If $\mathcal{R}$ is complete, C is the whole of $\mathbb{R} \times B$.

For (i) the action of G on $T^*\mathbb{R}^N$ is the cotangent lift of the orthogonal action of G on $\mathbb{R}^N$. For (ii) the contact action on $\mathbb{R} \times T^*\mathbb{R}^N$ is the cotangent lift of the orthogonal action of G on $\mathbb{R}^N$ to $T^*\mathbb{R}^N$ combined with the trivial action on $\mathbb{R}$.

Remark 4.9. If in lemma 4.7 the Reeb vector field $\mathcal{R}$ of (M, η) is complete, the announced result can also be deduced from proposition 4.1, lemma 4.6 and the construction used in [7] and [8] (in the case $\lambda = \infty$ a slight modification is needed).

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